Solution to Homework 1

Question 1.1

The household problem is given by:

$$\max_{C_s,I_s,L_s,K_{s+1}} U = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\log(C_s) + \chi_0 \frac{(1-L_s)^{1-\chi} - 1}{1-\chi} \right]$$

subject to the constraints that

$$\gamma_t : (1-\delta)K_t + I_t = K_{t+1}$$

$$\lambda_t : r_t K_t + w_t L_t = C_t + I_t$$

The necessary conditions for an equlibrium are listed below.

FOC wrt consumption from utility maximization

$$\frac{\partial}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0$$

$$\Leftrightarrow \lambda_t = \frac{1}{C_t} \tag{1}$$

FOC wrt labor from utility maximization

$$\frac{\partial}{\partial L_t} = -\chi_0 (1-L_t)^{-\chi} + \lambda_t w_t = 0$$

$$\Leftrightarrow \chi_0 (1-L_t)^{-\chi} = \lambda_t w_t \tag{2}$$

FOC wrt investment from utility maximization

$$\frac{\partial}{\partial I_t} = \gamma_t - \lambda_t = 0$$

$$\Leftrightarrow \gamma_t = \lambda_t \tag{3}$$

FOC wrt capital from utility maximization

$$\frac{\partial}{\partial K_t} = (1-\delta)\beta\gamma_{t+1} - \gamma_t + \beta\lambda_{t+1} r_{t+1} = 0 \tag{4}$$

FOC wrt γ_t

$$K_{t+1} = (1-\delta)K_t + I_t \tag{5}$$

FOC wrt λ_t

$$r_t K_t + w_t L_t = C_t + I_t \tag{6}$$

Question 1.2

The profit maximization conditions give us:

- Real wage rate

$$\begin{aligned} w_t &= \frac{\partial Y_t}{\partial L_t} \\ \frac{\partial Y_t}{\partial L_t} &= \left(\frac{Y_t}{L_t e^{N_t}}\right)^{\theta} e^{N_t} \end{aligned} \tag{7}$$

- Rental rate of capital

$$\begin{aligned} r_t &= \frac{\partial Y_t}{\partial K_t} \\ \frac{\partial Y_t}{\partial K_t} &= \nu \left(\frac{Y_t}{K_t e^{M_t}}\right)^{\theta} e^{M_t} \end{aligned} \tag{8}$$

- Technology constraint

$$Y_t = \left[\nu \left(K_t e^{M_t}\right)^{1-\theta} + \left(L_t e^{N_t}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{9}$$

Question 1.3

The equilibrium conditions are the FOCs from question 1.1, the FOCs from question 1.2, the technology constraint from question 1.2, the resource constraint below, and the two shock processes for the labor- and capital-augmenting productivity.

Resource constraint

$$Y_t = C_t + I_t \tag{10}$$

Question 1.4

See the matlab file *rbclab.m*.

Question 1.5

Combining equations (1), (2), and (7):

$$\chi_0 \left(1 - L\right)^{-\chi} = \frac{1}{C_t} \left(\frac{Y_t}{L_t}\right)^{\theta}. \tag{11}$$

Using the law of motion for capital,

$$\frac{I}{K} = \delta. \tag{12}$$

From equations (1), (3), and (4) we get

$$\begin{aligned} &(1-\delta)\beta\gamma - \gamma + \beta\gamma r = 0 \\ &\Leftrightarrow r = -1 + \delta + \frac{1}{\beta}. \end{aligned} \tag{13}$$

From equation (5), we can see that

$$S = \frac{I}{Y} = \delta \frac{K}{Y}. \tag{14}$$

From equation (8) we have that

$$\begin{aligned} &r = \nu \left(\frac{Y}{K}\right)^{\theta} \\ &\frac{r}{\nu} = \left(\frac{Y}{K}\right)^{\theta} \\ &\left(\frac{r}{\nu}\right)^{\frac{1}{\theta}} = \frac{Y}{K} \\ &\frac{K}{Y} = \left(\frac{\nu}{r}\right)^{\frac{1}{\theta}}. \end{aligned} \tag{15}$$

Combining equations (12) and (15)

$$S \equiv \frac{I}{Y} = \delta \left(\frac{\nu}{r}\right)^{\frac{1}{\theta}}.$$

Using equation (11) and $C = \left(1 - S\right) Y$

$$\begin{aligned} &\Leftrightarrow \chi_0 (1 - L)^{-\chi} = \frac{1}{(1 - S)\tilde{Y}} \left(\frac{Y}{L}\right)^{\theta} \\ &\Leftrightarrow \chi_0 (1 - L)^{-\chi} = \frac{1}{(1 - S)} \left(\frac{1}{L}\right)^{\theta} Y^{\theta - 1} \\ &\Leftrightarrow Y = \left[\chi_0 \left(1 - L\right)^{-\chi} \left(1 - S\right) L^{\theta}\right]^{\frac{1}{\theta - 1}}. \end{aligned} \tag{16}$$

From equation (9), we have

$$\tilde{Y} = \left[\nu\left(K\right)^{1-\theta} + \left(L\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{17}$$

Using equation (14) one can see that: $K = SY\frac{1}{\delta}$. Substituting into equation (17)

$$\begin{aligned}
&Y = \left[\nu\left(\frac{SY}{(\mu_N - 1 + \delta)}\right)^{1-\theta} + L^{1-\theta}\right]^{\frac{1}{1-\theta}} \\
&\Leftrightarrow Y^{1-\theta} = \nu\left(Y\right)^{1-\theta}\left(\frac{S}{\delta}\right)^{1-\theta} + L^{1-\theta} \\
&\Leftrightarrow Y = \left[1 - \nu\left(\frac{S}{\delta}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} L. \qquad (18)
\end{aligned}$$

Combining equations (16) and (18) we get:

$$\begin{aligned}
&\left[\chi_0\left(1-L\right)^{-\chi}\left(1-S\right)L^{\theta}\right]^{\frac{1}{\theta-1}} = \left[1 - \nu\left(\frac{S}{\delta}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} L \qquad (19) \\
&\Leftrightarrow \chi_0 = \frac{(1-L)^{\chi}}{L}\frac{1}{1-S}\left[1 - \nu\left(\frac{S}{\delta}\right)^{1-\theta}\right].
\end{aligned}$$

Question 1.6

Steady state capital payments as a share of output are given by $SHR_K \equiv r\frac{K}{Y}$. Using equations (13) and (15):

$$\begin{aligned}
&SHR_K \equiv r\frac{K}{Y} = \left(-1 + \delta + \frac{1}{\beta}\right)\left(\frac{\nu}{r}\right)^{\frac{1}{\theta}} \\
&\Leftrightarrow \left(SHR_K\right)^{\theta}\left(\frac{1}{\left(-1+\delta+\frac{1}{\beta}\right)}\right)^{\theta} = \frac{\nu}{r} \\
&\Leftrightarrow \nu = r\left(SHR_K\right)^{\theta}\left(\frac{1}{\left(-1+\delta+\frac{1}{\beta}\right)}\right)^{\theta} \\
&\Leftrightarrow \nu = \left(-1 + \delta + \frac{1}{\beta}\right)\left(SHR_K\right)^{\theta}\left(\frac{1}{\left(-1+\delta+\frac{1}{\beta}\right)}\right)^{\theta} \\
&\Leftrightarrow \nu = \left(SHR_K\right)^{\theta}\left(\frac{1}{-1+\delta+\frac{1}{\beta}}\right)^{\theta-1}. \qquad (20)
\end{aligned}$$

Question 1.7

See the matlab file *rbclab.m*.

For the second part of the question, the expression for χ_0 in terms of L^* that we derived above is strictly monotonic for $L^* \epsilon (0, 1]$. This can be easily seen taking the first derivative of χ_0 with respect to L^*. Notice that the term L^* in the derivative is always squared when $\chi = 2$, which means that the derivative cannot change sign keeping the other parameter values fixed. Hence, we have a one-to-one mapping between L^* and χ_0. In that case, a choice of L^* uniquely determines χ_0, and vice versa. Hence, chi_0 cannot influence the model dynamics other than through its implication for L^*.

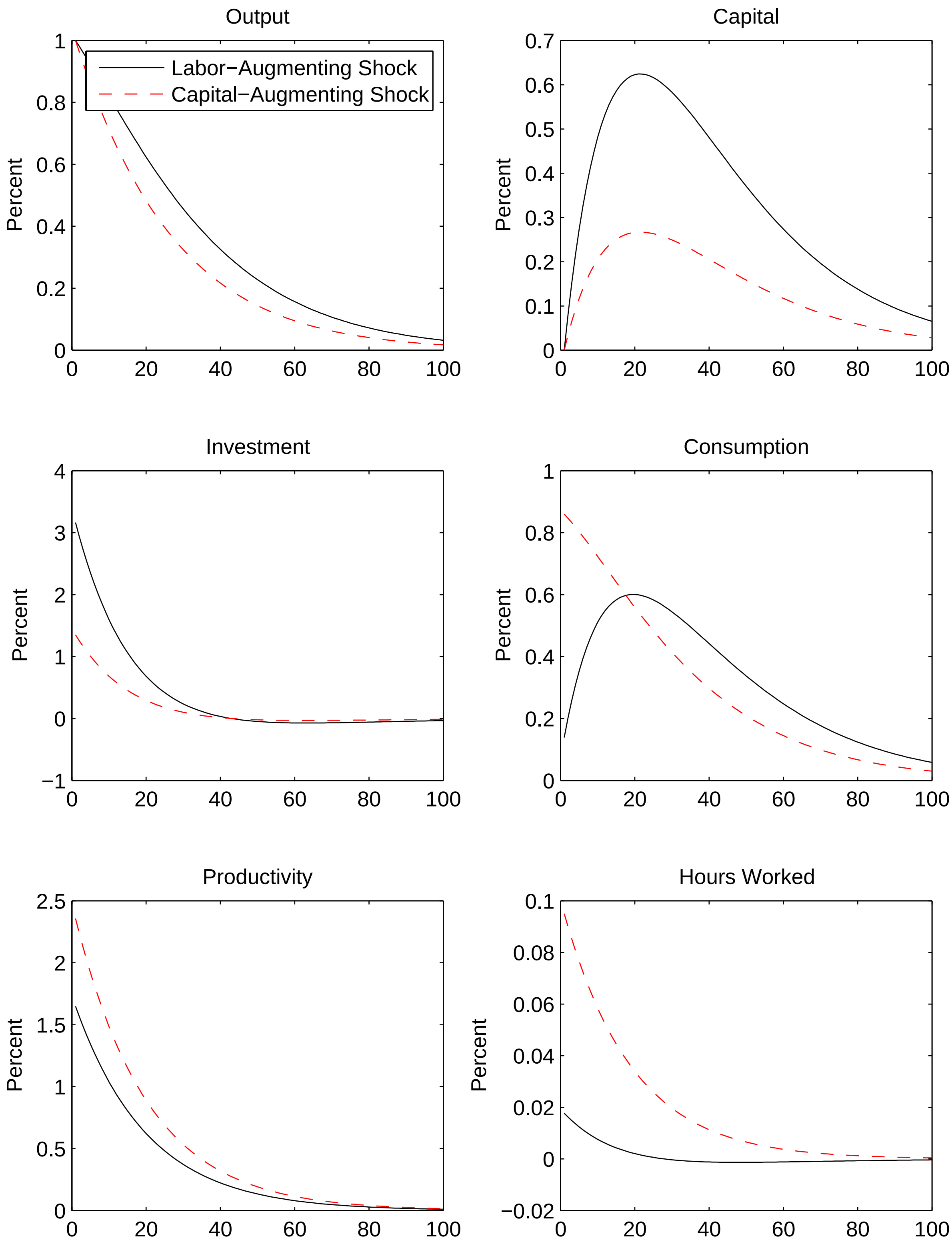
Question 1.8

See the matlab program *rbclab.m*

Question 1.9

The figure below, generated using *rbclab.m* compares capital- and labor-augmenting technology shocks. First of all, notice that our choice of $\theta = 2$ implies that capital and labor are complements, since their elasticity of substitution in production is 0.5. This implies that labor and the shock-augmented capital need to stay in relatively fixed proportions (the proportions would be exactly fixed in the limit of Leontiev technology when the elasticity of substitution is 0). The increase in capital-augmenting technology has a substitution effect that increases the marginal product of capital. This effect tends to push up investment. However, in order to produce at the technology frontier, we also have to push up labor to keep the factor proportions relatively constant. The technology shock also has a positive wealth effect, but both leisure and consumption have a positive wealth elasticity. Accordingly, in equilibrium, the rise in investment is muted, because to take advantage of the extra capital, labor would have to rise a suboptimal amount. Therefore, much of the increase in production is used to increase consumption, so that labor only rises modestly.

When we introduce a labor-augmenting technology shock, the response of investment is substantially different relative to the case discussed above. For a labor-augmenting shock, our choice of elasticity of substitution is going to imply that we need to push up the capital stock in order to remain at the efficient production frontier. Hence, much of the direct increase in production from the impact of the shock is going to go to increase investment, rather than consumption, at first. In a closed economy, the saving rate will increase much more substantially, relative to the case of the capital-augmenting shock.



Output
Labor−Augmenting Shock
Capital−Augmenting Shock
Capital
Investment
Consumption
Productivity
Hours Worked
Percent